

Geometry between the Folds

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Abstract

The purpose of this paper is to describe the mathematics that emanates from the construction of an origami box. We first construct a simple origami box and then discuss some of the mathematical questions that arise in the context of geometry and calculus.

The Common Core State Standards put a tremendous amount of emphasis on conceptual understanding (CCSSI, 2010). Origami provides a powerful context for conceptual understanding of mathematical ideas. Among other things, origami gives our students ready-made manipulatives that can be used to visualize abstract mathematical ideas in a concrete manner (Haga, 2006; Hull, 2006). For instance, when one creates a box from a square sheet of paper, the box becomes the object that can be manipulated and analyzed, and abstract concepts like length, width, height, volume, and surface area become something that one can touch. When students have objects that they have created to talk about, students communicate better with one another and with their teacher. Origami, when used in Non-Asian cultures, can also help students appreciate cultural diversity. Due to the link between origami and art, origami can additionally be used to inspire artistic-minded students to think mathematically. Lastly, origami creates a powerful context for the application of Howard Gardner's theory of Multiple Intelligences (Gardner, 2006; Wares, 2013). Gardner's theory of Multiple Intelligences incorporates several other dimensions of intelligences besides *linguistic* and *logical-mathematical* intelligence. Gardner identified the following six other intelligences: *bodily-kinesthetic intelligence*, *spatial intelligence*, *musical intelligence*, *interpersonal intelligence*, *intrapersonal intelligence*, and *naturalist intelligence* (Gardner, 2006).

When students have objects that they have created to talk about, students communicate better with one another and with their teacher.

In this paper, we learn to fold an origami box and discuss the mathematics embedded in the box. No experience in origami is needed to construct this box. However, it is important to make the creases sharp and accurate. Figure 1 illustrates the two types of creases that are formed when a piece of paper is folded. We need a square sheet of paper to fold the box. Let us use the following thirteen steps to construct the box. Each step corresponds to a picture in Figure 3.

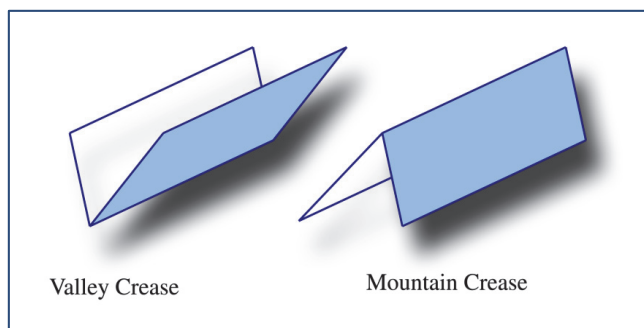


Figure 1. Valley crease and mountain crease.

Thirteen Steps to Make an Origami Box

1. Start with a square sheet with the plain side facing up.
2. Make a horizontal and a vertical crease that split the square into four congruent parts.
3. Make two more horizontal creases so that the three horizontal creases split the original square into four congruent rectangles.
4. Make two more vertical creases so that the three vertical creases split the original square into four congruent rectangles.
5. Drop vertex A_1 on vertex A_2 to form a slant crease as shown. Bring Vertex A_1 back to its original position. Repeat the process with the following pairs of vertices: B_1 on B_2 , C_1 on C_2 and D_1 on D_2 to form three more slant creases.
6. Flip the paper over.
7. Drop vertex P_1 on vertex P_2 to form another slant crease as shown. Repeat the process with the following pairs of vertices: Q_1 on Q_2 , R_1 on R_2 and S_1 on S_2 to form three more slant creases.
8. Make *concave* or *valley creases* only along the line segments that are marked with dots.
9. Flip the paper over.
10. Pull the four corners up as shown.
11. Press the extra flap against the four walls of the box that is about to be constructed.
12. At each of the four corners, there is a triangular pocket and a flap. Each flap will fit gently in the triangular pocket.
13. Your box is now complete.

Directions can also be found at <http://youtu.be/YTo1RXC2-bo>

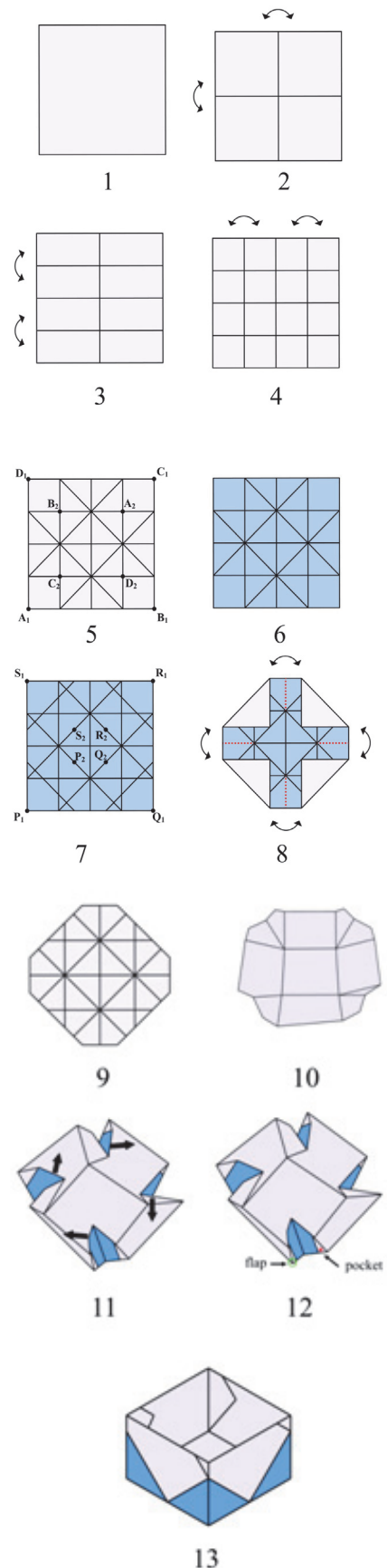


Figure 2. Pictures of steps 1-13

Let us carefully observe our creation. Having done that let us carefully open the box that we have created. Figure 3 shows the crease patterns created by the folds on the square sheet of paper. Note the shaded square in the center forms the square base of the constructed box and the four shaded rectangles around the base are the four 'walls' of the box. Let us assume the length of the original square sheet is a .

We can use the properties of right isosceles triangles to show the following relationship: $VQ = \frac{\sqrt{2}(3a)}{4}$.

By using the properties of right isosceles triangles, we can find the length of the base of the constructed box as $TS = \frac{\sqrt{2}a}{4}$.

By using the properties of right isosceles triangle, we can find the height of the constructed box as

$$\frac{1}{4}(VQ) = \frac{1}{4} \frac{\sqrt{2}(3a)}{4} = \frac{\sqrt{2}(3a)}{16}.$$

Now that we have the height and base of the box, we can conclude that the volume of the box is

$$\frac{\sqrt{2}(3a)}{16} \left(\frac{\sqrt{2}a}{4} \right)^2 = \frac{3\sqrt{2}a^3}{128}.$$

The author believes the activity of folding this origami box, and the mathematics described above, is appropriate for high-school geometry students. However, the same activity, with slight modifications, can be used to inspire calculus students to think mathematically. Now we discuss how this simple origami activity can be augmented to inspire students to use ideas of calculus.

Note when we folded the original box, $\frac{VC}{DC} = \frac{3}{4}$ (see Figure 3).

However, this ratio can vary. In fact, the ratio $\frac{VC}{DC}$ could be any real number greater than 0.5 and less than 1. Figure 4 illustrates a case where $\frac{VC}{DC} = \frac{4}{5}$. Note that in order to construct a box that will create crease patterns like the ones shown in Figure 4, we will need a ruler to make sure $\frac{VC}{DC} = \frac{4}{5}$. Once we measure the locations of points V, T, S, Q, M, P, W , and Y using a ruler, we can construct this new box by using steps that are similar to the steps that we used to construct the original box according to Figure 2. Even though rulers are not used in pure origami, we can break some of the rules just for the sake of asking more mathematically rich questions.

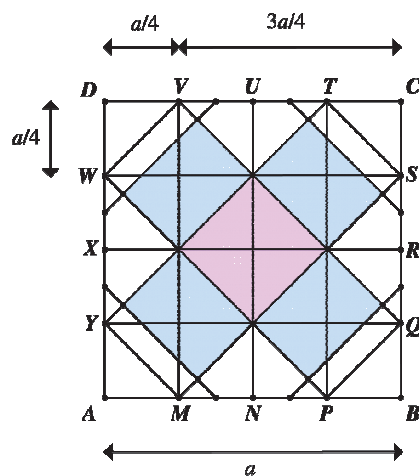


Figure 3. The crease marks created by the folds on the square paper.

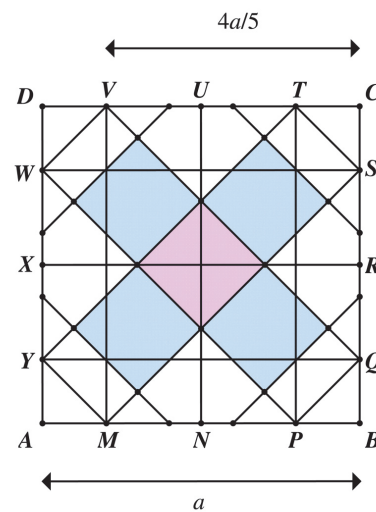


Figure 4. Crease marks of the box with $VC/DC = 4/5$.

Let us now analyze a more general case, the case where $\frac{VC}{DC}=x$, where $0.5 < x < 1$. Note the box in Figure 5 will not form for values of x equal to 0.5 and 1. Moreover, the folding becomes practically impossible when x is slightly more than 0.5 or slightly below 1.

We can use mathematical reasoning similar to the one we used earlier (when we found the volume of the original box) to conclude that the length of the square base of the box is $\sqrt{2}a(1-x)$, and the height of the box is $\frac{\sqrt{2}ax}{4}$. Therefore, the volume of the box is

$$V(x) = \frac{\sqrt{2}a^3x(1-x)^2}{2} = \frac{\sqrt{2}a^3(x^3 - 2x^2 + x)}{2}$$

where a is a constant, and $0.5 < x < 1$.

One of the questions that comes to mind is as follows: what value of x will help us construct the box with the largest volume? By using elementary calculus, we can conclude the following:

$$V'(x) = \frac{\sqrt{2}a^3(3x^2 - 4x + 1)}{2} = \frac{a^3(3x^2 - 4x + 1)}{\sqrt{2}}$$

$$\begin{aligned} V'(x) &= 0 \\ \Rightarrow \frac{a^3(3x^2 - 4x + 1)}{\sqrt{2}} &= 0 \\ \Rightarrow 3x^2 - 4x + 1 &= 0 \\ \Rightarrow (x-1)(3x-1) &= 0 \\ \Rightarrow x &= 1/3, \text{ or } 1 \end{aligned}$$

Note that both values are outside the domain of the function (0.5, 1). Also observe that $x^3 - 2x^2 + x = 0$, when x is 0 or 1. Since $V'(0.5) < 0$, we can conclude that the $V(x)$ is strictly decreasing over the interval (0.5, 1). That is, as x increases, $V(x)$ decreases. Figure 6 illustrates a sketch of $V(x)$ for $a = 4$.

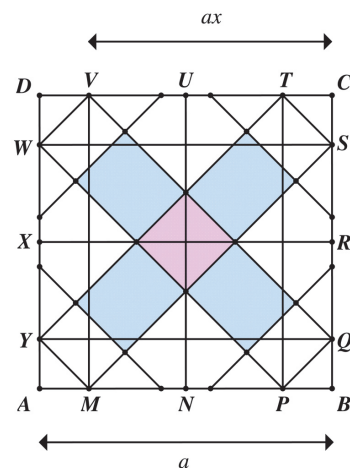


Figure 5. Crease marks of the box with $\frac{VC}{DC} = x$, with $0.5 < x < 1$.

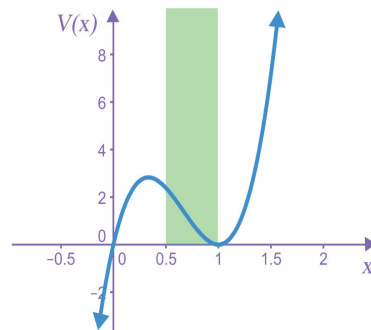


Figure 6. $V(x)$ when the value of $a = 4$.

Another interesting question that one may ask is “For what value or values of x would the constructed box be a cube?” To find out the answer, we set up the height of the box to be equal to the length of the base of the box and solve for x . In other words, we solve the following equation.

$$\begin{aligned}\sqrt{2}a(1-x) &= \frac{\sqrt{2}ax}{4} \\ \Rightarrow (1-x) &= \frac{x}{4} \\ \Rightarrow x &= \frac{4}{5}\end{aligned}$$

Therefore, when the value of x is $4/5$ the constructed box will be a cube without a lid.

The author used this activity as a part of high-school geometry courses and college-level geometry courses for pre-service teachers. This activity was well-received by the students in both types of courses. Students seem to get really excited when they use the power of their minds to physically turn a square sheet of paper into a box. The author also found the mathematical discussions that evolve during this activity to be relatively rich. The author conjectures that this may be due to the fact that students have something that they can physically manipulate (that is, the square sheet with the crease marks and the constructed box), so they find many of the related mathematical concepts to be more concrete and less abstract. The physical nature of the activity also helps students communicate their ideas better.



References

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